EXERCISE - 1.1

1. Find the sum $\frac{1}{5\times7}+\frac{1}{7\times9}+\frac{1}{9\times11}+\frac{1}{11\times13}+\frac{1}{13\times15}$.

2. Find the sum $\frac{1}{10}+\frac{1}{40}+\frac{1}{88}+\frac{1}{154}+\frac{1}{238}$.

3. Find the sum $\frac{1}{1+1^2+1^4}+\frac{2}{1+2^2+2^4}+\frac{3}{1+3^2+3^4}+\ldots+\frac{50}{1+50^2+50^4}$.

4. Find the sum 11 + 192 + 1993 + 19994 + 199995 + 1999996 + 19999997 + 199999998 + 1999999999.

5. Evaluate the expression $\frac{1^2}{1^2-10+50}+\frac{2^2}{2^2-20+50}+\ldots+\frac{9^2}{9^2-90+50}$.

6. Evaluate $\left(\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{2009}\right)\left(1+\frac{1}{2}+\ldots+\frac{1}{2008}\right)-\left(1+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{2009}\right)\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2008}\right)$.

7. Find the sum $\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+\ldots+51}$.

8. Evaluate $1^2-2^2+3^2-4^2+\ldots-2008^2+2009^2$.

9. Calculate $\frac{3^2+1}{3^2-1}+\frac{5^2+1}{5^2-1}+\frac{7^2+1}{7^2-1}+\ldots+\frac{99^2+1}{99^2-1}$.

10. Evaluate $\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\ldots+\frac{1}{100\times101\times102}$.

11. If x, y and z are positive integers such that $27x + 28y + 29z = 363$, find the value of $(x + y + z)$.

12. If the largest positive integer n such that $\sqrt{n-100}+\sqrt{n+100}$ is a rational number. Find the value of $\sqrt{n-1}$

13. The sum of $\frac{1}{2\times3\times4}+\frac{1}{3\times4\times5}+\frac{1}{4\times5\times6}+\ldots+\frac{1}{13\times14\times15}+\frac{1}{14\times15\times16}$ is $\frac{m}{n}$ in its lowest terms. Find the value of $n - 5m$.

14. Find the value of a + b where $\frac{1}{a!}-\frac{1}{b!}=\frac{3}{1!+2!+3!}+\frac{4}{2!+3!+4!}+\ldots+\frac{22}{20!+21!+22!}$.

EXERCISE - 1.2

1. Given $3x^2 + x = 1$, find the value of $6x^3 - x^2 - 3x + 100$.

2. Given $a^4 + a^3 + a^2 + a + 1 = 0$. Find the value of $a^{2000} + a^{2010} + 1$.

3. If $(x^2 - x - 1)^n = a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \ldots + a_1x^2 + a_1x + a_0$, find the value of $a_0 + a_2 + a_4 + \ldots + a_{2n}$.

4. If $x = \frac{a}{b+c} = \frac{b}{a+c} = \frac{c}{a+b}$, then find the value of x.

5. Find a natural number n, such that $2^8 + 2^{10} + 2^n$ is a perfect square number.

6. Given that f (x) is a polynomial of degree 3, and its remainders are $2x - 5$ and $-3x + 4$ when divided by $x^2 - 1$ and $x^2 - 4$ respectively. Find the f (–3).

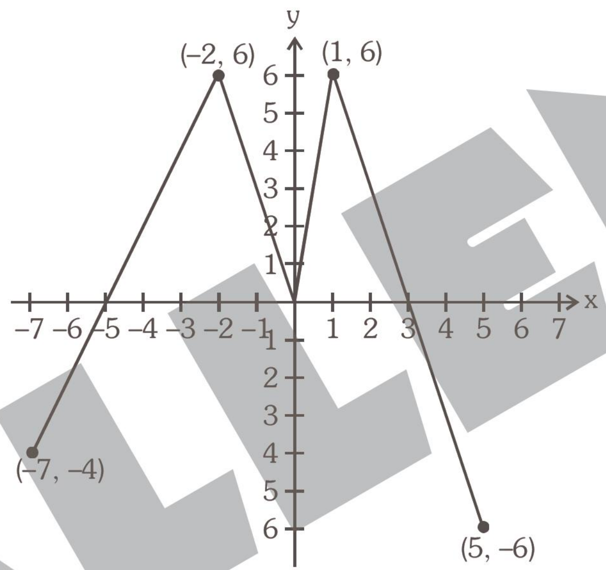
7. Factorize $x^4 + y^4 + (x + y)^4$.

8. Given that $f(x) = x^2 + ax + b$ is a polynomial with integral coefficients. If f is a common factor of polynomials $g(x) = x^4 - 3x^3 + 2x^2 - 3x + 1$ and $h(x) = 3x^4 - 9x^3 + 2x^2 + 3x - 1$, find f (x).

9. For any non-negative integers m, n, p, prove that the polynomial $x^{3m} + x^{3n+1} + x^{3p+2}$ has the factor $x^2 + x + 1$.

10. When $f(x) = x^3 + 2x^2 + 3x + 2$ is divided by g(x) which is a polyniomial with integer coefficients, the quotient and remainder are both h(x). Given that h is not a constant, find g and h.

11. The graph of the function f is shown below. How many solutions does the equation f(f(x)) = 6 have?



12. Suppose $x - y = 1$. Find the value of

$$x^4 - xy^3 - x^3y - 3x^2y + 3xy^2 + y^4.$$

13. If $x^2 + x - 1 = 0$, find the value of $x^4 - 3x^2 + 3$.

14. If two positive integers m and n, both bigger than 1, satisfy the equation

$$2005^2 + m^2 = 2004^2 + n^2,$$

find the value of $m + n - 200$.

15. Find an integer x that satisfies the equation

$$x^5 - 101x^3 - 999x^2 + 100900 = 0.$$

16. Let x and y be real numbers such that

$$x^2 + y^2 = 2x - 2y + 2.$$

What is the largest possible value of $x^2 + y^2 - \sqrt{32}$?

17. It is given that $x = \frac{1}{2-\sqrt{3}}$. Find the value of

$$x^6 - 2\sqrt{3}x^5 - x^4 + x^3 - 4x^2 + 2x - \sqrt{3}.$$

18. Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where a_i are nonnegative integers for $i = 0, 1, 2, \ldots, n$. If $f(1) = 21$ and $f(25) = 78357$, find the value of $\frac{f(10)+3}{100}$.

19. Let $m \neq n$ be two real numbers such that $m^2 = n + 2$ and $n^2 = m + 2$. Find the value of $4mn - m^3 - n^3$.

20. There are a few integer values of a such that $\frac{a^2-3a-3}{a-2}$ is an integer. Find the sum of all these integer values of a.

21. Suppose that a, b, x and y are real numbers such that

$$ax + by = 3,\ ax^2 + by^2 = 7,\ ax^3 + by^3 = 16 \quad \text{and} \quad ax^4 + by^4 = 42.$$

Find the value of $ax^5 + by^5$.

22. Let a and b be positive real numbers such that

$$\frac{1}{a} - \frac{1}{b} - \frac{1}{a+b} = 0.$$

Find the value of $\left(\frac{b}{a}+\frac{a}{b}\right)^2$.

23. Let the sum of the coefficients of the polynomial is S. Find $\frac{S}{16}$.

$$(4x^2 - 4x + 3)^4 (4 + 3x - 3x^2)^2.$$

24. If

$$f(x) = \frac{1}{\sqrt[3]{x^2+2x+1} + \sqrt[3]{x^2-1} + \sqrt[3]{x^2-2x+1}}$$

for all positive integers x, find the value of

$$f(1) + f(3) + f(5) + \cdots + f(997) + f(999).$$

25. Let a and b be two integers. Suppose $x^2 - x - 1$ is a factor of the polynomial $ax^5 + bx^4 + 1$. Find the value of a.

26. Suppose f is a function satisfying $f(x + x^{-1}) = x^6 + x^{-6}$, for all $x \neq 0$. Determine $400 - f(3)$.

27. Suppose x_1, x_2 and x_3 are roots of $(11 - x)^3 + (13 - x)^3 = (24 - 2x)^3$. What is the sum of $x_1 + x_2 + x_3$?

28. Suppose that $a + x^2 = 2006$, $b + x^2 = 2007$ and $c + x^2 = 2008$ and $abc = 3$. Find the value of

$$\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}.$$

29. Find the value of

$$\frac{x^4 - 6x^3 - 2x^2 + 18x + 23}{x^2 - 8x + 15} \text{ when } x = \sqrt{19 - 8\sqrt{3}}.$$

30. Let x, y and z be three real numbers such that $xy + yz + xz = 4$. Find the least possible value of $x^2 + y^2 + z^2$.

31. Consider polynomials P(x) of degree at most 3, each of whose coefficients is an element of {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}. If total k polynomials satisfy $P(-1) = -9$, then find $k - 200$?

32. For certain real numbers a, b, and c, the polynomial $g(x) = x^3 + ax^2 + x + 10$ has three distinct roots, and each root of g(x) is also a root of the polynomial $f(x) = x^4 + x^3 + bx^2 + 100x + c$. What is $\frac{-f(1)}{91}$?

33. Let P be a cubic polynomial with $P(0) = k$, $P(1) = 2k$, and $P(-1) = 3k$. If $P(2) + P(-2) = JK$. What is $J \times 5$?

34. Let $a > 0$, and let P(x) be a polynomial with integer coefficients such that

$P(1) = P(3) = P(5) = P(7) = a$, and

$P(2) = P(4) = P(6) = P(8) = -a$.

The smallest possible value of $a = 5 \times Q$. What is Q ?

35. Find the remainder when $x^{2008} + 2008x + 2008$ is divided by $x + 1$.

36. Given that x and y are positive real numbers such that $(x + y)^2 = 2500$ and $xy = 500$, find the exact value of $\frac{x^3 + y^3}{10^3}$.

37. Find the remainder when $(x - 1)^{100} + (x - 2)^{200}$ is divided by $x^2 - 3x + 2$.

38. Let p(x) be a polynomial with real coefficients such that for all real x,

$$2(1 + p(x)) = p(x - 1) + p\{x + 1)$$

and $p(0) = 8$, $p(2) = 32$. Determine the value of p(4).

39. Let a, b, c, d, e be five numbers satisfying the following conditions:

$a + b + c + d + e = 0$, and

$abc + abd + abe + acd + ace + ade + bcd + bce + bde + cde = 33$.

Find the value of $\frac{a^3 + b^3 + c^3 + d^3 + e^3}{502}$.

40. Given that x and y are both negative integers satisfying the equation $y = \frac{10x}{10 - x}$, find the maximum value of $y + 20$.

41. Given that x and y are real numbers satisfying the following equations :

$$x + xy + y = 2 + 3\sqrt{2} \text{ and } x^2 + y^2 = 6,$$

then the value of $|x + y + 1| = a + \sqrt{b}$. Find the product of a and b.

42. Given that $y = (x - 16)(x - 14)(x + 14)(x + 16)$, find the maximum value of $(-0.1y)$.

43. Given that $a + \frac{1}{a+1} = b + \frac{1}{b-1} - 2$ and $a - b + 2 \neq 0$, find the value of $ab - a + b$.

44. If $|x| + x + 5y = 2$ and $|y| - y + x = 7$, find the value of $(x + y)^3$.

45. Let x_1, x_2, x_3, x_4 denote the four roots of the equation

$$x^4 + kx^2 + 90x - 2009 = 0.$$

If $x_1x_2 = 49$, find the value of k.

46. Let x be a real number such that $x^2 - 15x + 1 = 0$. Find the last two digits of $x^4 + \frac{1}{x^4}$.

47. If x, y and z are real numbers such that $x + y + z = 9$ and $xy + yz + zx = 24$, find the largest possible value of z.

48. It is given that $\sqrt{a} - \sqrt{b} = 20$, where a and b are real numbers. Find the maximum possible value of $\left(\frac{a - 5b}{10}\right)$.

49. Given that $169(157 - 77x)^2 + 100(201 - 100x)^2 = 26(77x - 157)(1000x - 2010)$, find the value of x.

50. Evaluate $\frac{(2020^2 - 20100)(20100^2 - 100^2)(2000^2 + 20100)}{10(2010^6 - 10^6)}$.

51. If a, b, c and d are real numbers such that

$$\frac{b+c+d}{a} = \frac{a+c+d}{b} = \frac{a+b+d}{c} = \frac{a+b+c}{d} = r,$$

find the sum of all the possible values of r.

52. Find the value of x that satisfies the equation $25^{-2} = \frac{5^{48/x}}{5^{26/x} \cdot 25^{17/x}}$.

53. Let the value(s) of x is such that $8xy - 12y + 2x - 3 = 0$ is true for all values of y. Then find (16x).

54. If p, q and r are distinct roots of $x^3 - x^2 + x - 2 = 0$, then $p^3 + q^3 + r^3$ equals

55. Let $g(x) = x^5 + x^4 + x^3 + x^2 + x + 1$. What is the remainder when the polynomial $g(x^{12})$ is divided by the polynomial g(x)?

56. Find the number of pairs (m, n) of integers which satisfy the equation $m^3 + 6m^2 + 5m = 27n^3 + 9n^2 + 9n + 1$.

57. How many real numbers x satisfy the equation $3^{2x+2} - 3^{x+3} - 3^x + 3 = 0$?

58. If $y + 4 = (x - 2)^2$, $x + 4 = (y - 2)^2$, and $x \neq y$, what is the value of $x^2 + y^2$?

59. For how many integers is the number $x^4 - 51x^2 + 50$ negative?

60. What is the sum of all the roots of the equation

$\sqrt{5|x|+8} = \sqrt{x^2 - 16}$.

EXERCISE - 1.3

1. For what values of b do the equations:
$1988x^2 + bx + 8891 = 0$ and $8891x^2 + bx + 1988 = 0$ have a common root?

2. Given that the equation in x has at least a real root, find the range of m.
$(m^2 - 1)x^2 - 2(m + 2)x + 1 = 0$

3. If the equation is x has real roots, then find the value of a and b.
$x^2 + 2(1 + a)x + (3a^2 + 4ab + 4b^2 + 2) = 0$

4. Mr. Fat is going to pick three non-zero real number and Mr. Taf is going to arrange the three numbers as the coefficients of a quadratic equation.

$$\square x^2 + \square x + 0 = 0$$

Mr. Fat wins the game if and only if the resulting equation has two distinct rational solutions. Who has a winning strategy?

5. a, b, c are three distinct non-zero real numbers. Prove that the following three equations $ax^2 + 2bx + c = 0$, $bx^2 + 2cx + a = 0$, and $cx^2 + 2ax + b = 0$ cannot all have two equal real roots.

6. If $x^2 + x + 1 = 0$, find the value of $x^{1999} + x^{2000}$.

7. For $x^2 + 2x + 5$ to be a factor of $x^4 + px^2 + q$, find the values of p and q.

8. If $a + b + c = 0$, find $\dfrac{b^2 + c^2 + a^2}{b^2 - ca}$.

9. For how many real values of a will
$x^2 + 2ax + 2008 = 0$ has two integer roots ?

10. If x, y are positive real numbers satisfying the system of equations
$x^2 + y\sqrt{xy} = 336$, $y^2 + x\sqrt{xy} = 112$, then x + y equals

11. a, b, c are positive integers such that $a^2 + 2b^2 - 2bc = 100$ and $2ab - c^2 = 100$. Then $\dfrac{a+b}{c}$ is

12. When x is real, the greatest possible value of $10^x - 100^x$ is

13. Find integers 'a' and 'b' such that $(x^2 - x - 1)$ divides $ax^{17} + bx^{16} + 1$.

14. Find the real points (x, y) satisfying $3x^2 + 3y^2 - 4xy + 10x - 10y + 10 = 0$.

15. Solve for x, y and z ; if $xy + x + y = 23$, $yz + y + z = 31$, $zx + z + x = 47$.

16. α, β are the real roots of the equation $x^2 - px + q = 0$. Find the number of the pairs (p, q) such that the quadratic equation with roots α^2, β^2 is still $x^2 - px + q = 0$.

17. Given that a and b are the real roots $x^2 - 2x - 1 = 0$, find the value of $5\alpha^4 + 12\beta^3$.

18. Given that the real numbers s, t satisfy $19s^2 + 99s + 1 = 0$, $t^2 + 99t + 19 = 0$, and $st \neq 1$. Find the value of $\dfrac{st + 4s + 1}{t}$.

19. Given that $a = 8 - b$ and $c^2 = ab - 16$, prove that a = b.

20. Given that a, b are roots of the equation $x^2 - 7x + 8 = 0$, where $a > b$. Find the value of $\dfrac{2}{\alpha} + 3\beta^2$ without solving the equation.

21. Both roots of the quadratic equation $x^2 - 63x + k = 0$ are prime numbers. The number of possible values of k is

22. Find the sum of all possible values of a such that the following equation has real root in x:

$$(x - a)^2 + (x^2 - 3x + 2)^2 = 0.$$

23. Let p be a real number such that the equation $2y^2 - 8y = p$ has only one solution.
Then find the value of p + 64.

24. Let a, b and c be the lengths of the three sides of a triangle. Suppose a and b are the roots of the equation

$$x^2 + 4(c + 2) = (c + 4)x,$$

and the largest angle of the triangle is x°. Find the value of x.

25. How many ordered pairs of integers (x, y) satisfy the equation

$$x^2 + y^2 = 2(x + y) + xy?$$

26. Let n be a positive integer such that one of the roots of the quadratic equation

$$4x^2 - (4\sqrt{3} + 4)x + \sqrt{3}n - 24 = 0$$

is an integer. Find the value of n.

27. Suppose that the two roots of the equation

$$\frac{1}{x^2 - 10x - 29} + \frac{1}{x^2 - 10x - 45} - \frac{2}{x^2 - 10x - 69} = 0$$

are α and β. Find the value of $(-\alpha\beta)$.

28. Find total number of pairs of (x, y) where x, y belongs to integer that satisfy the equation

$$x + y = x^2 - xy + y^2.$$

29. Let a and b be two integers. Suppose that $\sqrt{7 - 4\sqrt{3}}$ is a root of the equation $x^2 + ax + b = 0$. Find the value of $b - a$.

30. Let p be an integer such that both roots of the equation

$$5x^2 - 5px + (66p - 1) = 0$$

are positive integers. Find the value of p.

31. Suppose a and b are the roots of $x^2 + x \sin\alpha + 1 = 0$ while c and d are the roots of the equation $x^2 + x\cos\alpha - 1 = 0$. Find the value of $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}$.

32. The zeroes of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of all possible values of a?

33. Let a, b, and c be three distinct one-digit numbers. The maximum value of the sum of the roots of the equation $(x - a)(x - b) + (x - b)(x - c) = 0$ is k. What is the value of 2k?

34. There are exactly N distinct rational numbers k such that $|k| < 200$ and $5x^2 + kx + 12 = 0$ has at least one integer solution for x. What is N?

35. Let a and b be the roots of $x^2 + 2000x + 1 = 0$ and let c and d be the roots of $x^2 - 2008x + 1 = 0$. Now P is defined as $P = (a + c)(b + c)(a - d)(b - d)$. Then find sum of all the digits of P

36. Find the value of the smallest positive integer m such that the equation

$$x^2 + 2(m + 5)x + (100m + 9) = 0$$

has only integer solutions.

37. Given that $(m - 2)$ is a positive integer and it is also a factor of $3m^2 - 2m + 10$. Find the sum of all such values of m.

38. Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + (1/b)$ and $b + (1/a)$ are the roots of the equation $x^2 - px + q = 0$. What is 2q?

39. The quadratic equation $x^2 + mx + n = 0$ has roots that are twice those of $x^2 + px + m = 0$, and none of m, n, and p is zero. What is the value of $\frac{n}{p}$?

40. What is the negative of the sum of the reciprocals of the roots of the equation

$$\frac{2003}{2004}x+1+\frac{1}{x}=0?$$

41. Both roots of the quadratic equation $x^2 - 63x + k = 0$ are prime numbers. The number of possible values of k is

42. Suppose that a and b are are nonzero real numbers, and that the equation $x^2 + ax + b = 0$ has solutions a and b. Find the value of (a – b).

43. Let a, b, and c be real numbers such that $a - 7b + 8c = 4$ and $8a + 4b - c = 7$. Then $a^2 - b^2 + c^2$ is

44. The zeroes of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of all possible values of a?

EXERCISE - 1.4

1. Solve the equation $\frac{x-n}{m}-\frac{x-m}{n}=\frac{m}{n}$ (where $mn \neq 0$).

2. Given that – 2 is the solution of equation $\frac{1}{3}mx = 5x + (-2)^2$, find the value of the expression $(m^2 - 11m + 17)^{2007}$.

3. Solve the equation $[4ax - (a + b)](a + b) = 0$, where a and b are constants.

4. If x and y can take only natural number values, find the number of (x, y) pairs satisfying the equation $2x + 5y = 100$.

5. If a, b c are real numbers such that $a+\left(\frac{1}{b}\right)=\frac{7}{3}$; $b+\left(\frac{1}{c}\right)=4$; $c+\left(\frac{1}{a}\right)=1$, find abc.

6. If a, b, c, d, satisfy the equations $a + 7b + 3c + 5d = 0$, $8a + 4b + 6c + 2d = -16$, $2a + 6b + 4c + 8d = 16$, $5a + 3b + 7c + d = -16$ then the value of $(a + d)(b + c) =$

7. If $x + y = 5xy$, $y + z = 6yz$, $z + x = 7zx$ find the value of $x + y + z$.

8. Given that the equations in x :

$$3\left[x-2\left(x+\frac{a}{3}\right)\right] = 2x \text{ and } \frac{3x+a}{3}-\frac{1+4x}{6}=0$$

have a common solution. Find the common solution.

9. If positive numbers a, b, c satisfy abc = 1, solve the equation in x

$$\frac{2ax}{ab+a+1}+\frac{2bx}{bc+b+1}+\frac{2cx}{ca+c+1}=1.$$

10. If $a_{n+1}=\frac{1}{1+\frac{1}{a_n}}$ (n = 1, 2, ..., 2008) and $a_1 = 1$, find the value of $a_1a_2 + a_2a_3 + a_3a_4 + \ldots + a_{2008}a_{2009}$.

11. Solve the system of equations

$$\begin{cases} x-y-z = 5 \\ y-z-x = 1 \\ z-x-y = -15 \end{cases}$$

12. Solve the system of equations

$$\begin{cases} x-y+z &= 1 \\ y-z+u &= 2 \\ z-u+v &= 3 \\ u-v+x &= 4 \\ v-x+y &= 5 \end{cases}$$

13. Given

$$\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=0, \qquad \ldots(i)$$

$$\frac{1}{x}-\frac{6}{y}-\frac{5}{z}=0 \qquad \ldots(ii)$$

Find the value of $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$.

14. Solve the system of equations

$$\begin{cases} \frac{1}{x}+\frac{1}{y+z}=\frac{1}{2} \\ \frac{1}{y}+\frac{1}{z+x}=\frac{1}{3} \\ \frac{1}{z}+\frac{1}{x+y}=\frac{1}{4} \end{cases}$$

15. Let f be a function with the following properties:

(i) $f(1) = 1$, and

(ii) $f(2n) = n \times f(n) \cdot$ for any positive integer n.

If the value of $f(2^{100}) = 2^{49xy}$, then the digits x and y respectively?

16. Find the sum of all the real numbers x that satisfy the equation

$$(3^x - 27)^2 + (5^x - 625)^2 = (3^x + 5^x - 652)^2.$$

17. Find the largest real number x such that

$$\frac{x^2+x-1+|x^2-(x-1)|}{2} = 35x-250.$$

18. There are n balls in a box, and the balls are numbered 1, 2, 3,... , n respectively. One of the balls is removed from the box, and it turns out that the sum of the numbers on the remaining balls in the box is 5048. If the number on the ball removed from the box is m, find the value of m.

19. The total number of integers between 0 and 10^5 having the digit sum equal to 8 is n then find $5(500 - n)$.

20. Define a function on the positive integers recursively by $f(1) = 2$, $f(n) = f(n-1) + 1$ if n is even, and $f(n) = f(n-2) + 2$ if n is odd and greater than 1. What is $f(2017) - 2000$?

21. Suppose that x and y are nonzero real numbers such that $\frac{3x+y}{x-3y} = -2$. What is the value of $\left(\frac{x+3y}{3x-y}\right)^4$?

22. Danica drove her new car on a trip for a whole number of hours, averaging 55 miles per hour. At the beginning of the trip, abc miles were displayed on the odometer, where abc is a 3-digit number with $a \geq 1$ and $a + b + c \leq 7$. At the end of the trip, where the odometer showed cba miles. What is $a^2 + b^2 + c^2$?

23. Given that x and y distinct nonzero real numbers such that $x + \frac{2}{x} = y + \frac{2}{y}$, what is $45xy - 45$?

24. Real numbers x and y satisfy the equation $x^2 + y^2 = 10x - 6y - 34$. What is $(x + y)^4$?

25. Let a, b, and c be real numbers such that $a + b + c = 2$, and $a^2 + b^2 + c^2 = 12$

The difference between the maximum and minimum possible values of c is $\frac{p}{q}$ with coprime p and q.

What is $(p + q^2)$?

26. Let a, b, and c be positive integers with $a \geq b \geq c$ such that $a^2 - b^2 - c^2 + ab = 2011$ and

$a^2 + 3b^2 + 3c^2 - 3ab - 2ac - 2bc = -1997$. What is $a \div 11$?

27. Let a, b, c, d and e be positive integers with $a + b + c + d + e = 2010$, and let M be the largest of the sums $a + b$, $b + c$, $c + d$, and $d + e$. The smallest possible value of $M = 11 \times p$. What is p ?

28. Suppose that $f(x + 3) = 3x^2 + 7x + 4$ and $f(x) = ax^2 + bx + c$. What is $(a + b + c)^4$?

29. For each positive integer n, let $f(n) = n^4 - 360n^2 + 400$. The sum of all values of f(n) that are prime numbers is a three digit number pqr. What is p + r ?

30. Let a, b, c, d, and e be distinct integers such that $(6 - a)(6 - b)(6 - c)(6 - d)(6 - e) = 45$. What is $a + b + c + d + e$?

31. Let m, n be integers such that $1 < m \leq n$. Define

$$f(m,n) = \left(1-\frac{1}{m}\right)\times\left(1-\frac{1}{m+1}\right)\times\left(1-\frac{1}{m+2}\right)\times\ldots\times\left(1-\frac{1}{n}\right).$$

If $S = f(2, 2049) + f(3, 2049) + f(4, 2049) + \ldots + f(2049, 2049)$, find the value of $\frac{S}{64}$.

32. The number of positive integral solutions (a, b, c, d) satisfying $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=1$

with the condition that $a < b < c < d$ is

33. For integers $a_1,\ldots, a_n \in \{1, 2, 3,\ldots, 9\}$, we use the notation $\overline{a_1a_2\ldots a_n}$ to denote the number $10^{n-1} a_1 + 10^{n-2} a_2 + \ldots + 10a_{n-1} + a_n$. For example, when a = 2 and b = 0, $\overline{ab}$ denotes the number 20. Given that $\overline{ab} = b^2$ and $\overline{acbc} = (\overline{ba})^2$. Find the value of a+b+c.

34. It is known that there is only one pair of positive integers a and b such that $a \leq b$ and $a^2 + b^2 + 8ab = 2010$. Find the value of a + b.

35. Let $f(x) = \frac{x^{2010}}{x^{2010}+(1-x)^{2010}}$. Find the value of

$$\frac{f\left(\frac{1}{2011}\right)+f\left(\frac{2}{2011}\right)+f\left(\frac{3}{2011}\right)+\ldots+f\left(\frac{2010}{2011}\right)}{15}$$

36. If

$$x + \sqrt{x^2 - 1} + \frac{1}{x - \sqrt{x^2 - 1}} = 20$$

then if $$x^2 + \sqrt{x^4 - 1} + \frac{1}{x^2 + \sqrt{x^4 - 1}} = k, \text{ find } [k]$$

37. If x and y non-zero real numbers such that

$|x| + y = 3$

and $|x|\, y + x^3 = 0$,

then find $|x - y|$

38. How many distinct ordered triples (x, y, z) satisfy the equations

$x + 2y + 4z = 12$

$xy + 4yz + 2xz = 22$

$xyz = 6$

39. How many ordered pairs of real numbers (x, y) satisfy the following system of equations?

$$x + 3y = 3$$

$$||x| - |y|| = 1$$

40. What is the sum of all the solutions of $x = |2x - |60 - 2x||$?

41. On a 50-question multiple choice math contest, students receive 4 points for a correct answer, 0 points for an answer left blank, and –1 point for an incorrect answer. Jasvinder's total score on the contest was 99. What is the maximum number of questions that Jasvinder could have answered correctly?

EXERCISE - 1.5

1. Tarun and Mayank run in opposite directions on a circular track, starting at diametrically opposite points. They first meet after Tarun has run 100 meters. They next meet after Mayank has run 150 meters past their first meeting point. Each girl runs at a constant speed. What is the length of the track in meters after dividing it by 10?

2. The two digits in Sunil's age are the same as the digits in Ramesh's age, but in reverse order. In five years Sunil will be twice as old as Ramesh will be then. What is the difference in their current ages?

3. Driving at a constant speed, Kavita usually takes 180 minutes to drive from her house to her mother's house. One day Kavita begins the drive at her usual speed, but after driving $\frac{1}{3}$ of the way, she hits a bad snowstorm and reduces her speed by 20 km per hour. This time the trip takes her a total of 276 minutes. Kavita is now H kms far from her mother's house. What is $\frac{H}{5}$?

4. At kota Summer Camp, 60% of the children play soccer, 30% of the children swim, and 40% of the soccer players swim. To the nearest whole percent, what percent of the non-swimmers play soccer?

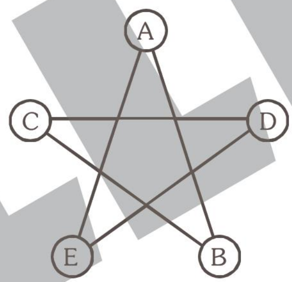
5. Amit and Rahul are 20 kilometers apart. They bike toward one another with Amit traveling three times as fast as Rahul, and the distance between them decreasing at a rate of 1 kilometer per minute. After 5 minutes, Amit stops biking because of a flat tire and waits for Rahul. After how many minutes from the time they started to bike does Rahul reach Amit?

6. Yesterday Ram drove 1 hour longer than Shyam at an average speed 5 miles per hour faster than Shyam. Akash drove 2 hours longer than Shyam at an average speed 10 miles per hour faster than Shyam. Ram drove 70 miles more than Shyam. If Akash drove x km more than Shyam then find $\frac{x}{10}$

7. The saxena family consists of a mother, a father, and some children. The average age of the members of the family is 20, the father is 48 years old, and the average age of the mother and children is 16. How many children are in the family?

8. A teacher gave a test to a class in which 10% of the students are juniors and 90% are seniors. The average score on the test was 84. The juniors all received the same score, and the average score of the seniors was 83. What score did each of the juniors receive on the test?

9. In the five-sided star shown, the letters A, B, C, D and E are replaced by the numbers 3, 5, 6, 7, and 9 although not necessarily in this order. The sums of the numbers at the ends of the line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$ and $\overline{EA}$ form an arithmetic sequence, although not necessarily in this order. What is the middle term of the arithmetic sequence?



10. A man bought two paintings and then sold them for 300 rupees each. He made a profit of 20% for the first painting, but a loss of 20% for the second painting. State the amount of profit or loss in rupees.

11. Each valve A, B, and C, when open, releases water into a tank as it own constant rate. With all three valves open, the tank fills in 1 hour, with only valves A and C open it takes 1.5 hour, and with only valves B and C open it takes 2 hours. The number of minutes required with only valves A and B open is

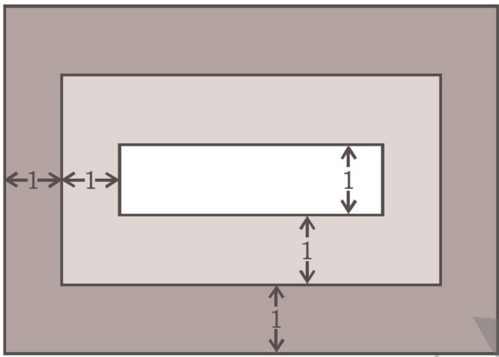
12. Mohit has a collection of 23 coins, consisting of 5-paise coins, 10-paise coins, and 25-paise coins. He has 3 more 10-paise coins than 5-paise coins, and the total value of his collection is 320 paise. How many more 25-paise coins does Mohit have than 5-paise coins?

13. Sameer drove 96 km in 90 minutes. His average speed during the first 30 minutes was 60 kph (kilometer per hour), and his average speed during the second 30 minutes was 65 kph. What was his average speed, in kph, during the last 30 minutes?

14. Every week Ramesh pays for a movie ticket and a soda out of his allowance. Last week, Ramesh's allowance was A Rupees. The cost of his movie ticket was 20% of the difference between A and the cost of his soda, while the cost of his soda was 5% of the difference between A and the cost of his movie ticket. To the nearest whole percent, what fraction of A did Ramesh pay for his movie ticket and soda?

15. Sania set off on her bicycle to visit her friend, traveling at an average speed of 17 kilometers per hour. When she had gone half the distance to her friend's house, a tire went flat, and she walked the rest of the way at 5 kilometers per hour. In all it took her 44 minutes to reach her friend's house. If Sania had to walk for x kilometer. Find the value of 6x.

16. A rug is made with three different colors as shown. The areas of the three differently colored regions form an arithmetic progression. The inner rectangle is one foot wide, and each of the two shaded regions is 1 foot wide on all four sides. What is the length in feet of the inner rectangle?



17. Two years ago Pankaj was three times as old as his cousin Naveen. Two years before that, Pankaj was four times as old as Naveen. In how many years will the ratio of their ages be 2 : 1?

18. Disha has 12 coins, each of which is a 5-cent coin or a 10-cent coin. There are exactly 17 different values that can be obtained as combinations of one or more of her coins. How many 10-cent coins does Disha have?

19. Consider the set of all fractions $\frac{x}{y}$, where x and y are relatively prime positive integers. How many of these fractions have the property that if both numerator and denominator are increased by 1, the value of the fraction is increased by 10%?

20. Suppose that 4 cows give 10 gallons of milk in 2 days. At this rate, how many gallons of milk will 5 cows give in 4 days?

21. Hari drives from his home to the airport to catch a flight. He drives 35 km in the first hour, but realizes that he will be 1 hour late if he continues at this speed. He increases his speed by 15 km per hour for the rest of the way to the airport and arrives 30 minutes early. If the distance between airport and his home is P km. Find $\frac{P}{7}$.

22. Divya drove her new car on a trip for a whole number of hours, averaging 55 km per hour. At the beginning of the trip, abc km were displayed on the odometer, where abc is a 3-digit number with $a \geq 1$ and $a + b + c \leq 7$. At the end of the trip, where the odometer showed cba km. What is $a^2 + b^2 + c^2$?

23. A basketball team's players were successful on 50% of their two-point shots and 40% of their three-point shots, which resulted in 54 points. They attempted 50% more two-point shots than three-point shots. How many three-point shots did they attempt?

24. It takes yuvi 60 seconds to walk down an escalator when it is not operating and only 24 seconds to walk down the escalator when it is operating. How many seconds does it take yuvi to ride down the operating escalator when she just stands on it?

25. Kareena walks once around a track at exactly the same constant speed every day. The sides of the track are straight, and the ends are semicircles. The track has width 6 meters, and it takes her 36 seconds longer to walk around the outside edge of the track than around the inside edge. If Kareena's speed in meters per second is $\frac{\pi}{k}$, find k.

EXERCISE - 1.6

1. Suppose that

$\log_2(\log_3(\log_5(\log_7 N))) = 11.$

How many different prime numbers are factors of N?

(1) 1 (2) 2 (3) 3 (4) 4

2. Suppose that

$\log_2(\log_2(\log_2 x)) = 2.$

How many digits are in the base-10 representation for x?

(1) 5 (2) 7 (3) 9 (4) 11

3. How many positive integers b have the property that $\log_b 729$ is also a positive integer?

(1) 1 (2) 2 (3) 3 (4) 4

4. Let $f(n) = \log_{2002} n^2$ for all positive integers n. Define

$N = f(11) + f(13) + f(14).$

Which of the following relations is true?

(1) $N > 1$ (2) $N = 1$ (3) $1 < N < 2$ (4) $N = 2$

5. For some real numbers a and b, the equation

$8x^3 + 4ax^2 + 2bx + a = 0$

has three distinct positive roots, and the sum of the base-2 logarithms of the roots is 5. What is the value of a?

(1) – 256 (2) – 64 (3) –8 (4) 64

6. For any positive integer n, define

$$f(n) = \begin{cases} \log_8 n, & \text{if } \log_8 n \text{ is rational,} \\ 0, & \text{otherwise.} \end{cases}$$

What is $\sum_{n=1}^{1997} f(n)$?

(1) $\log_8 2047$ (2) 6 (3) $\frac{55}{3}$ (4) $\frac{58}{3}$

7. What is the value of the expression

$$N = \frac{1}{\log_2 100!} + \frac{1}{\log_3 100!} + \frac{1}{\log_4 100!} + \ldots + \frac{1}{\log_{100} 100!}?$$

(1) 0.01 (2) 0.1 (3) 1 (4) 2

8. What is the value of the sum

$S = \log_{10}(\tan 1°) + \log_{10}(\tan 2°) + \ldots + \log_{10}(\tan 88°) + \log_{10}(\tan 89°)$?

(1) 0 (2) $\frac{1}{2}\log_{10}\left(\frac{1}{2}\sqrt{3}\right)$ (3) $\frac{1}{2}\log_{10} 2$ (4) $\frac{1}{2}\log_{10} 3$

9. Let $a \geq b > 1$. What is the largest possible value of

$\log_a \frac{a}{b} + \log_b \frac{b}{a}$?

(1) – 2 (2) 0 (3) 2 (4) 3

10. The set of all real numbers x for which $\log_{2004}(\log_{2003}(\log_{2002}(\log_{2001} x)))$ is defined is $\{x \mid x > c\}$. If value of c is in the form $(200x)^{200y}$, then the value of xy

11. For some real numbers a and b, the equation

$$8x^3 + 4ax^2 + 2bx + a = 0$$

has three distinct positive roots. If the sum of the base-2 logarithms of the roots is 5, what is the value of $\frac{-a}{8}$?

12. If $\log(xy^3) = 1$ and $\log(x^2y) = 1$, If $\log(xy) = \frac{a}{b}$ then a + b = ?

13. Evaluate

$$\frac{1}{\log_2 12\sqrt{5}} + \frac{1}{\log_3 12\sqrt{5}} + \frac{1}{\log_4 12\sqrt{5}} + \frac{1}{\log_5 12\sqrt{5}} + \frac{1}{\log_6 12\sqrt{5}}.$$

14. The solutions to the equation $\log_{3x} 4 = \log_{2x} 8$, where x is a positive real number other than $\frac{1}{3}$ or $\frac{1}{2}$, can be written as $\frac{p}{q}$ where p and q are relatively prime positive integers. What is p + q?

15. The value of $\log_3 7 . \log_5 9 . \log_7 11 . \log_9 13 \ldots . \log_{21} 25 . \log_{23} 27$ is x. Find x^2.

16. The value of $(625^{\log_5 2015})^{\frac{1}{4}}$ is a four digit number A. What is $\frac{A}{31}$?

17. When $p = \sum_{k=1}^{6} k \ln k$, the number e^p is an integer. The largest power of 2 that is a factor of e^p is 2^R. What is R?

18. There are q values of x which satisfy $\log_{10}(x - 40) + \log_{10}(60 - x) < 2$. What is the value of q ?

19. Let m > 1 and n > 1 be integers. Suppose that the product of the solutions for x of the equation

$8(\log_n x)(\log_m x) - 7\log_n x - 6\log_m x - 2013 = 0$ is the smallest possible integer. What is m + n?

20. The solution of the equation $7^{x+7} = 8^x$ can be expressed in the form $x = \log_b 7^7$. What is 21b?

21. x = abc (a 3-digit number) satisfy the equation

$\log_{\sqrt{2}} \sqrt{x} + \log_2 x + \log_4 (x^2) + \log_8(x^3) + \log_{16}(x^4) = 40$. What is ab (a 2 digit number)?

22. There are t distinct four-tuples (a, b, c, d) of rational numbers with $a\log_{10}2 + b\log_{10}3 + c\log_{10}5 + d\log_{10}7 = 2005$. What is $25 \times t$?

23. Let S be the set of ordered triples (x, y, z) of real numbers for which $\log_{10}(x + y) = z$ and $\log_{10}(x^2 + y^2) = z + 1$.

There are real numbers a and b such that for all ordered triples (x, y, z) in S we have $x^3 + y^3 = a.\ 10^{3z} + b.10^{2z}$. What is the value of 4(a + b)?

24. Find the number of positive integers x that satisfy the inequality $\left|3 + \log_x \frac{1}{3}\right| < \frac{8}{3}$.

25. Suppose that a, b and c are real numbers greater than 1. Find the value of

$$\frac{1}{1+\log_{a^2b}\left(\frac{c}{a}\right)} + \frac{1}{1+\log_{b^2c}\left(\frac{a}{b}\right)} + \frac{1}{1+\log_{c^2a}\left(\frac{b}{c}\right)}.$$

26. Find the number of positive integers x, where $x \neq 9$, such that

$$\log_{\frac{x}{9}}\left(\frac{x^2}{3}\right) < 6 + \log_3\left(\frac{9}{x}\right).$$

Where $x \in (1,100)$

27. If $a > b > 1$ and $\frac{1}{\log_a b} + \frac{1}{\log_b a} = \sqrt{1229}$, find the value of $\frac{1}{\log_{ab} b} - \frac{1}{\log_{ab} a}$.

28. If $x, y > 0$, $\log_y x + \log_x y = \frac{10}{3}$ and $xy = 144$, then $\frac{x+y}{2\sqrt{3}}$ is

29. If $b > 1$, $x > 0$ and $(2x)^{\log_b 2} - (3x)^{\log_b 3} = 0$, then $\frac{1}{x}$ is

EXERCISE - 1.7

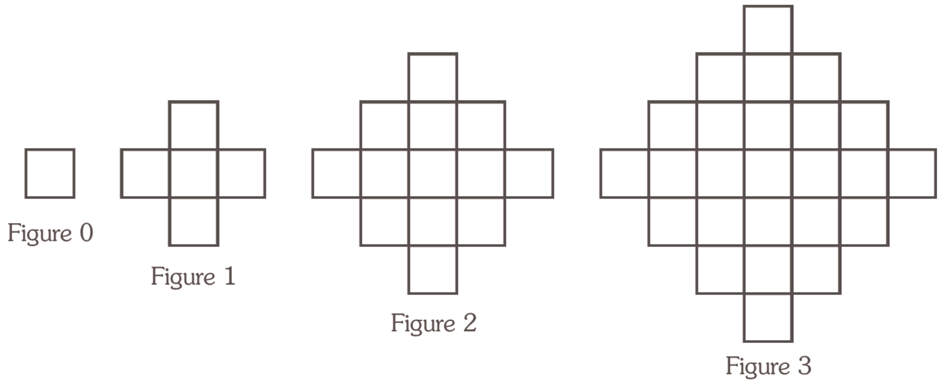
1. A grocer makes a display of cans in which the top row has one can and each lower row has two more cans than the row above it. If the display contains 100 cans, how many rows are there?

(1) 5 (2) 8 (3) 9 (4) 10

2. The second and fourth terms of a geometric sequence are 2 and 6. Which of the following is a possible first term?

(1) $-\sqrt{3}$ (2) $-\frac{2\sqrt{3}}{3}$ (3) $-\frac{\sqrt{3}}{3}$ (4) $\sqrt{3}$

3. Figures 0, 1, 2, and 3 consists of 1, 5, 13, and 25 nonoverlapping unit squares, respectively. If the pattern were continued, how many non-overlapping unit squares would there be in Figure 100?



(1) 10,401 (2) 19,801 (3) 20,201 (4) 39,801

4. Let 1, 4, . . . and 9, 16, . . . be two arithmetic sequences. The set S is the union of the first 2004 terms of each sequence. How many distinct numbers are in S?

(1) 3722 (2) 3732 (3) 3914 (4) 3924

5. Let $a_1, a_2, \ldots, a_k$ be a finite arithmetic sequence with

$$a_4 + a_7 + a_{10} = 17,$$

$a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} = 77$,

and $a_k = 13$. What is k?

(1) 16 (2) 18 (3) 20 (4) 22 (5) 24

6. A sequence of three real numbers forms an arithmetic sequence whose first term is 9. If the first term is unchanged, 2 is added to the second term, and 20 is added to the third term, then the three resulting numbers form a geometric sequence. What is the smallest possible value for the third term of the geometric progression?

(1) 1 (2) 4 (3) 36 (4) 49

7. Consider the sequence of numbers: 4, 7, 1, 8, 9, 7, 6, ... For n > 2, the nth term of the sequence is the units digit of the sum of the two previous terms. Let S_n denote the sum of the first n terms of this sequence. What is the smallest value of n for which $S_n > 10{,}000$?

(1) 1992 (2) 1999 (3) 2001 (4) 2002

8. Suppose that the sequence $\{a_n\}$ is defined by

$$a_1 = 2, \text{ and } a_{n+1} = a_n + 2n, \text{ when } n \geq 1.$$

What is a_{100}?

(1) 9900 (2) 9902 (3) 9904 (4) 10100

9. The increasing sequence of positive integers $a_1, a_2, a_3, \ldots$ has the property that $a_{n+2} = a_n + a_{n+1}$, for all $n \geq 1$. Suppose that $a_7 = 120$. What is a_8?

(1) 128 (2) 168 (3) 193 (4) 194

10. A sequence of three real numbers forms an arithmetic progression with a first term of 9. If 2 is added to the second term and 20 is added to the third term, the three resulting numbers form a geometric progression. What is the largest possible value for the third term in the geometric progression?

11. In the sequence 2001, 2002, 2003, . . ., each term after the third is found by subtracting the previous term from the sum of the two terms that precede that term. For example, the fourth term is 2001 + 2002–2003 = 2000. What is the 2004th term in this sequence?

12. Consider the sequence of numbers: 4, 7, 1, 8, 9, 7, 6, ... For n > 2, the nth term of the sequence is the units digit of the sum of the two previous terms. Let Sn denote the sum of the first n terms of this sequence. If smallest value of n for Sn > 10, 000 is abcd then find a + b + c + d:

13. Let x and y be positive real numbers. What is the smallest possible value of $\frac{16}{x}+\frac{108}{y}$ + xy?

14. Find the value of the positive integer n if

$$\frac{1}{\sqrt{4}+\sqrt{5}}+\frac{1}{\sqrt{5}+\sqrt{6}}+\frac{1}{\sqrt{6}+\sqrt{7}}+\ldots+\frac{1}{\sqrt{n}+\sqrt{n+1}}=5.$$

15. Consider a sequence of real numbers $\{a_n\}$ defined by

$$a_1 = 1 \text{ and } a_{n+1} = \frac{a_n}{1+na_n} \text{ for } n \ge 1.$$

Find the value of $\frac{1}{a_{2005}} - 2009000$.

16. James calculates the sum of the first n positive integers and finds that the sum is 5053. If he has counted one integer twice, which one is it?

17. What is the value of

$$2100 - (x+1)(x+2006)\left[\frac{1}{(x-1)(x+2)}+\frac{1}{(x+2)(x+3)}+\ldots+\frac{1}{(x+2005)(x+2006)}\right]?$$

18. Suppose a_n denotes the last two digits of 7^n. For example, $a_2 = 49$, $a_3 = 43$. The value of $a_1 + a_2 + a_3 + \ldots\ldots\ldots + a_{2007}$ is given by x. Find sum of all the digits of x.

19. A sequence $\{a_n\}$ is defined by $a_1 = 2$, $a_n = \frac{1+a_{n-1}}{1-a_{n-1}}$, $n \ge 2$. Find the value of $(1100 + 2008\, a_{2007})$.

20. If the minimum value of $\sqrt{\sum_{k=1}^{100}|n-k|}$, where n ranges over all positive integers, is m, find $\frac{m}{50}$.

21. The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of S?

22. Let a < b < c be three integers such that a, b, c is an arithmetic progression and a, c, b is a geometric progression. What is the smallest possible value of 10c?

23. The sequence $S_1, S_2, S_3, \ldots\ldots\ldots S_{10}$ has the property that every term beginning with the third is the sum of the previous two. That is, $S_n = S_{n-2} + S_{n-1}$ for $n \ge 3$. Suppose that $S_9 = 110$ and $S_7 = 42$. What is S_4?

24. The sequence $\log_{12}162$, $\log_{12}x$, $\log_{12}y$, $\log_{12}z$, $\log_{12}1250$ is an arithmetic progression. What is $\frac{x}{10}$?

25. The internal angles of quadrilateral ABCD form an arithmetic progression. Triangles ABD and DCB are similar with $\angle DBA = \angle DCB$ and $\angle ADB = \angle CBD$. Moreover, the angles in each of these two triangles also form an arithmetic progression. In degrees, the largest possible sum of the two largest angles of ABCD is R. What is $\frac{R}{3}$?

26. Two non-decreasing sequences of nonnegative integers have different first terms. Each sequence has the property that each term beginning with the third is the sum of the previous two terms, and the seventh term of each sequence is N. The smallest possible value of N is n. What is half of n ?

27. The first four terms of an arithmetic sequence are p, 9, 3p – q, and 3p + q. What is the sum of digits of the 2010th term of the sequence?

28. Let $a + ar_1 + ar_1^2 + ar_1^3 \ldots\ldots$ and $a + ar_2 + ar_2^2 + ar_2^3 + \ldots\ldots$ be two different infinite geometric series of positive numbers with the same first term. The sum of the first series is r_1, and the sum of the second series is r_2. What is $31(r_1 + r_2)$?

29. For each positive integer n, the mean of the first n terms of a squence is n. What will be the answer when 2008th term of the sequence is divided by 55 ?

30. The geometric series a + ar + ar^2 + …. has a sum of 7, and the terms involving odd powers of r have a sum of 3. What is 10(a + r)?

31. Find the maximum value of $\sqrt{x-144}+\sqrt{722-x}$.

32. If $S=\sum_{k=1}^{99}\frac{(-1)^{k+1}}{\sqrt{k(k+1)}(\sqrt{k+1}-\sqrt{k})}$, find the value of 10S.

33. Let $a_1, a_2, a_3,\ldots$ be the sequence of all positive integers that are relatively prime to 75, where $a_1 < a_2 < a_3 < \cdots$. (The first five terms of the sequence are: $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $a_4 = 7$, $a_5 = 8$.)

Then the value of a_{2008} is defined as $p10^3 + q10^2 + r10 + s$. Find (p + q + r + s)

34. Given that

$$x + (1 + x)^2 + (1 + x)^3 + \ldots + (1 + x)^n = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n,$$

where each a_r is an integer, r = 0, 1, 2,…, n.

Find the value of n such that $a_0 + a_2 + a_3 + a_4 + \ldots + a_{n-2} + a_{n-1} = 60 - \frac{n(n+1)}{2}$.

35. Given that $a_{n+1}=\frac{a_{n-1}}{1+na_{n-1}a_n}$, where n = 1, 2, 3, … and $a_0 = a_1 = 1$, find the value of $\left(2\times10^4-\frac{1}{a_{199}a_{200}}\right)$.

36. Suppose that (u_n) is a sequence of real numbers satisfying $u_{n+2} = 2u_{n+1} + u_n$, and that $u_3 = 9$ and $u_6 = 128$. What is u_5?

37. For each positive integer n, the mean of the first n terms of a sequence is n. What is the square root of 2008th term of the sequence to the nearest integer?

38. A finite sequence of three-digit integers has the property that the tens and units digits of each terms are, respectively, the hundreds and tens digits of the next term, and the tens and units digits of the last term are, respectively, the hundreds and tens digits of the first term. For example, such a sequence might begin with terms 247, 475, and 756 and end with the term 824. Let S be the sum of all the terms in the sequence. What is the largest prime number that always divides S?

39. How many non-similar triangle have angles whose degree measures are distinct positive integers in arithmetic progression?

40. Let $a_1, a_2, \ldots$ be a sequence for which

$$a_1 = 2\ a_2 = 3 \text{ and } a_n = \frac{a_{n-1}}{a_{n-2}} \text{ for each positive integer } n \geq 3.$$

What is a_{2006}?

41. Let $a_1, a_2, \cdots$, be a sequence with the following properties.

I. $a_1 = 1$, and

II. $a_{2n} = n \cdot a_n$ for any positive integer n.

Find the sum of digits of 'n' if $a_{2^{100}} = 2^n$.

42. Let 1, 4, $\cdots$ and 9, 16, $\cdots$ be two arithmetic progressions. The set S is the union of the first 2004 terms of each sequence. Suppose there are 'n' distinct numbers in set 'S', then find the sum of the digits of 'n'.

43. Suppose that $\{a_n\}$ is an arithmetic sequence with $a_1 + a_2 + \cdots + a_{100} = 100$ and $a_{101} + a_{102} + \cdots + a_{200} = 200$. What is the value of $10^3(a_2 - a_1)$?

44. Let $\{a_k\}$ be a sequence of integers such that $a_1 = 1$ and $a_{m+n} = a_m + a_n + mn$, for all positive integers m and n. Then a_{12} is

45. Let $a_1, a_2, \ldots\ldots\ldots a_k$ be a finite arithemetic sequence with

$a_4 + a_7 + a_{10} = 17$ and $a_4 + a_5 + a_6 + \ldots\ldots + a_{12} + a_{13} + a_{14} = 77$.

If $a_k = 13$, the k =

46. Suppose x, y, z is a geometric sequence with common ratio r and $x \neq y$. If x, 2y, 3z is an arithemetic sequence, then, 27r is?

47. The sequence

$$1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, 2, 2, 2, 2, 1, 2, \ldots$$

consists of 1's separated by blocks of 2' s with n 2' s in the n^{th} block.

The sum of digits of the sum of the first 1234 terms of this sequence is

48. Define a sequence of real numbers $a_1, a_2, a_3, \ldots$ by $a_1 = 1$ and $a_{n+1}^3 = 99a_n^3$ for all $n \geq 1$. Then $(a_{100})^{1/33}$ equals

49. The number of terms in an A.P. (Arithmetic Progression) is even. The sums of the odd- and even-numbered terms are 24 and 30 respectively. If the last term exceeds the first by 10.5, the number of terms in the A.P. is

50. In a geometric series of positive terms the difference between the fifth and fourth terms is 576, and the difference between the second and first terms is 9. What is the last two digits of sum of the first five terms of this series?

51. Let $a_1, a_2, \ldots, a_{2018}$ be a strictly increasing sequence of positive integers such that

$$a_1 + a_2 + \cdots + a_{2018} = 2018^{2018}.$$

What is the remainder when $a_1^3 + a_2^3 + \ldots + a_{2018}^3$ is divided by 6?

52. A function f is defined recursively by f(1) = f(2) = 1 and f(n) = f(n – 1) – f(n – 2) + n for all integers $n \geq 3$. What is the value of [f(2018) – 2000]?

53. Define a sequence recursively by $F_0 = 0$, $F_1 = 1$, and F_n = the remainder when $F_{n-1} + F_{n-2}$ is divided by 3, for all $n \geq 2$. Thus the sequence starts 0, 1, 1, 2, 0, 2 . . .. What is $F_{2017} + F_{2018} + F_{2019} + F_{2020} + F_{2021} + F_{2022} + F_{2023} + F_{2024}$?

54. The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of S?

55. How many four-digit integers abcd, with $a \neq 0$, have the property that the three two-digit integers ab < bc < cd form an increasing arithmetic sequence? One such number is 4692, where a = 4, b = 6, c = 9, and d = 2.

56. The product (8)(888 . . . 8), where the second factor has k digits, is an integer whose digits have a sum of 1000. Find the value of (1000 – k) ?

57. Consider the set of numbers $\{1, 10, 10^2, 10^3, \ldots 10^{10}\}$. The ratio of the largest element of the set to the sum of the other ten elements of the set is closest to which integer?

EXERCISE - 1.8

1. The expression $4^4 \cdot 9^4 \cdot 4^9 \cdot 9^9$ simplifies to which of the following?

(1) 13^{13} (2) 13^{36} (3) 36^{13} (4) 36^{36}

2. The expression $\frac{15^{30}}{45^{15}}$ simplifies to which of the following?

(1) $\left(\frac{1}{3}\right)^2$ (2) 1 (3) 3^{15} (4) 5^{15}

3. What is the value of k if $2^{2007} - 2^{2006} - 2^{2005} + 2^{2004} = k \cdot 2^{2004}$?

(1) 1 (2) 2 (3) 3 (4) 4

4. Suppose that $x > y > 0$. Which of the following is the same as $\frac{x^y y^x}{y^y x^x}$?

(1) $(x - y)^{y/x}$ (2) $\left(\frac{x}{y}\right)^{x-y}$ (3) 1 (4) $\left(\frac{x}{y}\right)^{y-x}$

5. What is the value of $\sqrt{\frac{8^{10}+4^{10}}{8^4+4^{11}}}$?

(1) $\sqrt{2}$ (2) 16 (3) 32 (4) $12^{2/3}$

6. Let $f(x) = x^{(x+1)}(x+2)^{(x+3)}$. What is the value of

$$f(0) + f(-1) + f(-2) + f(-3)?$$

(1) 0 (2) $\frac{8}{9}$ (3) 1 (4) $\frac{10}{9}$

7. Which of the following values of x satisfies the expression

$$25^{-2} = \frac{5^{48/x}}{5^{26/x} \cdot 25^{17/x}}?$$

(1) 2 (2) 3 (3) 5 (4) 6

8. Suppose that $a > 0$ and $b > 0$. Define r to be the number that results when both the base and the exponent of a^b are tripled. Suppose now that we write $r = a^b \cdot x^b$. Which of the following expressions represents x?

(1) 3 (2) $3a^2$ (3) $27a^2$ (4) $2a^{3b}$

9. What is the sum of all the real numbers x that satisfy

$$(2^x - 4)^3 + (4^x - 2)^3 = (4x + 2^x - 6)^3?$$

(1) 2 (2) $\frac{5}{2}$ (3) 3 (D) $\frac{7}{2}$

10. Suppose that $60^a = 3$ and $60^b = 5$. What is the value of $12^{(1-a-b)/(2-2b)}$?

(1) $\sqrt{3}$ (2) 2 (3) $\sqrt{5}$ (4) 3

11. Let a_k be the coefficient of x^k in the expansion of $(l + 2x)^{100}$, where $0 \le k \le 100$. Find the number of integers $r : 0 \le r \le 99$ such that $a_r < a_{r+1}$.

12. Let a_k, be the coefficient of x^k in the expansion of

$$(x + 1) + (x + 1)^2 + (x + 1)^3 + (x + 1)^4 + \cdots + (x + 1)^{99}.$$

Determine the value of $[a_4/a_3]$.

13. Let $a_1, a_2, \ldots$ be a sequence of rational numbers such that $a_1 = 2$ and for $n \geq 1$

$$a_{n+1} = \frac{1+a_n}{1-a_n}.$$

Determine $30 \times a_{2008}$.

14. If x is real, such that $(2^x - 4)^3 + (4^x - 2)^3 = (4^x + 2^x - 6)^3$, if sum of all real x is $\frac{m}{n}$, where m and n co-prime, find $m + n$

15. For some particular value of N, when $(a + b + c + d + 1)^N$ is expanded and like terms are combined, the resulting expression contains exactly 1001 terms that include all four variables a, b, c, and d, each to some positive power. What is N?

EXERCISE - 1.9

1. Six numbers from a list of nine integers are 7, 8, 3, 5, 9, and 5. What is the largest possible value of the median of all nine numbers in this list?

(1) 5 (2) 6 (3) 7 (4) 8

2. Consider the sequence

$$1, -2, 3, -4, 5, -6, \ldots,$$

whose nth term is $(-1)^{n+1} \cdot n$. What is the average of the first 200 terms of the sequence?

(1) – 1 (2) – 0.5 (3) 0 (4) 0.5

3. A speaker talked for sixty minutes to a full auditorium. Twenty percent of the audience heard the entire talk and ten percent slept through the entire talk. Half of the remainder heard one third of the talk and the other half heard two thirds of the talk. What was the average number of minutes of the talk heard by members of the audience?

(1) 24 (2) 27 (3) 30 (4) 33

4. All the students in an algebra class took a 100-point test. Five students scored 100, each student scored at least 60, and the mean score was 76. What is the smallest possible number of students in the class?

(1) 10 (2) 11 (3) 12 (4) 13

5. In the sixth, seventh, eighth, and ninth basketball games of the season, a player scored 23, 14, 11, and 20 points, respectively. Her pointsper-game average was higher after nine games than it was after the first five games. Her average after ten games was greater than 18. What is the least number of points she could have scored in the tenth game?

(1) 26 (2) 27 (3) 28 (4) 29

6. The mean of three numbers is 10 more than the least of the numbers and 15 less than the greatest. The median of the three numbers is 5. What is their sum?

(1) 5 (2) 20 (3) 25 (4) 30

7. The mean, median, unique mode, and range of a collection of eight integers are all equal to 8. What is the largest integer that can be an element of this collection?

(1) 11 (2) 12 (3) 13 (4) 14

8. The sequence $a_1, a_2, a_3, \ldots$, satisfies $a_1 = 19$ and $a_9 = 99$. Also, a_n is the arithmetic mean of the first $n - 1$ terms whenever $n \geq 3$. What is a_2?

(1) 59 (2) 79 (3) 99 (4) 179

9. For positive integers m and n such that $m + 10 < n + 1$, both the mean and the median of the set $\{m, m + 4, m + 10, n + 1, n + 2, 2n\}$ are equal to n. What is $m + n$?

10. A list of 2018 positive integers has a unique mode, which occurs exactly 10 times. The least number of distinct values that can occur in the list is a three digit number xyz. What is $(y + z)^2$?

11. A, B, C are three piles of rocks. The mean weight of the rocks in A is 40 kg, the mean weight of the rocks in B is 50 kg, the mean weight of the rocks in the combined piles A and B is 43 kg, and the mean weight of the rocks in the combined piles A and C is 44 kg. What is the greatest possible integer value for the mean in kg of the rocks in the combined piles B and C?

12. On a certain math exam, 10% of the students got 70 points, 25% got 80 points, 20% got 85 points, 15% got 90 points, and the rest got 95 points. What is the difference between the mean and the median score on this exam?

13. The average (arithmetic mean) age of a group consisting of doctors and lawyers is 40. If the doctors average 35 and the lawyers 50 years old, then the ratio of the number of doctors to the number of lawyers is K : 1. Find K.

14. Last year Sunita took 7 math tests and received 7 different scores, each an integer between 91 and 100, inclusive. After each test she noticed that the average of her test scores was an integer. Her score on the seventh test was 95. If her score on the sixth test was S. Find $\frac{S}{4}$.

15. The mean, median, and mode of the 7 data values 60, 100, x, 40, 50, 200, 90 are all equal to x. What is the value of x?

ANSWER KEY

EXERCISE - 1.1

1. $\frac{1}{15}$ **2.** $\frac{5}{17}$ **3.** $\frac{1275}{2551}$ **4.** 2222222184 **5.** y **6.** $-\frac{2007}{4018}$

7. $\frac{25}{26}$ **8.** 2019045 **9.** $49\frac{49}{100}$ **10.** $\frac{2575}{10302}$ **11.** 13 **12.** 50

13. 95 **14.** 24

EXERCISE - 1.2

1. 99 **2.** 3 **3.** $\frac{1+(-1)^n}{2}=\begin{cases}1, n = \text{even}\\0, n = \text{odd}\end{cases}$ **4.** $\frac{1}{2}$ or -1 **5.** n = 10

6. 53 **7.** $2(x^2 + y^2 + xy)^2$ **8.** $f(x) = x^2 - 3x + 1$

10. $g(x) = x^2 + x + 1$, $h(x) = x + 1$ **11.** 06 **12.** 01 **13.** 02 **14.** 11

15. 10 **16.** 06 **17.** 02 **18.** 51 **19.** 00 **20.** 08 **21.** 20

22. 05 **23.** 81 **24.** 05 **25.** 03 **26.** 78 **27.** 36 **28.** 01

29. 05 **30.** 04 **31.** 20 **32.** 77 **33.** 70 **34.** 63 **35.** 01

36. 50 **37.** 01 **38.** 64 **39.** 99 **40.** 15 **41.** 06 **42.** 90

43. 02 **44.** 27 **45.** 07 **46.** 27 **47.** 05 **48.** 50 **49.** 31

50. 10 **51.** 02 **52.** 03 **53.** 24 **54.** 04 **55.** 06 **56.** 00

57. 02 **58.** 15 **59.** 12 **60.** 00

EXERCISE - 1.3

1. $b = \mp 10879$ **2.** $m \ge -\frac{5}{4}$ **3.** $a = 1$, $b = -\frac{1}{2}$

4. Mr. Fat has a winning strategy **6.** – 1 **7.** p = 6 , q = 25 **8.** 2

9. 8 **10.** 20 **11.** 2 **12.** $\frac{1}{4}$ **13.** a = 987 , b = – 1597

14. (x, y) = (–1, 1) **15.** (5, 3, 7) or (– 7, – 5, – 9) **16.** 3 pairs are possible : (0, 0), (1, 0) and (2, 1)

17. 169 **18.** – 5 **19.** a = b = 4 **20.** $\frac{403-85\sqrt{17}}{8}$ **21.** 01 **22.** 03

23. 56 **24.** 90 **25.** 06 **26.** 12 **27.** 39 **28.** 06 **29.** 05

30. 76 **31.** 01 **32.** 16 **33.** 33 **34.** 78 **35.** 15 **36.** 90

37. 51 **38.** 09 **39.** 08 **40.** 01 **41.** 01 **42.** 03 **43.** 01

44. 16

EXERCISE - 1.4

1. $x = \begin{cases} \dfrac{n^2}{n-m}, & n \neq m \\ \text{no sol}^n, & n = m \end{cases}$ **2.** -1 **3.** $x = \begin{cases} \text{if } a+b=0 \\ \text{Then no sol}^n, \\ \text{if } a+b \neq 0 \\ \text{Then } x = \dfrac{a+b}{a}, a \neq 0 \\ \qquad \text{no sol}^n, a = 0 \end{cases}$ **4.** 9

5. 1 **6.** – 16 **7.** $\frac{13}{12}$ **8.** $x = -\frac{1}{3}$ is a common solution **9.** $\frac{1}{2}$

10. $\frac{2008}{2009}$ **11.** x = 7, y = 5, z = –3 **12.** x = 0, y = 6, z = 7, u = 3, v = –1 **13.** – 1

14. $x = \frac{23}{10}, y = \frac{23}{6}, z = \frac{23}{2}$ **15.** 50 **16.** 07 **17.** 25 **18.** 02

19. 25 **20.** 18 **21.** 16 **22.** 37 **23.** 45 **24.** 16 **25.** 25
26. 23 **27.** 61 **28.** 16 **29.** 10 **30.** 25 **31.** 16 **32.** 06
33. 18 **34.** 42 **35.** 67 **36.** 51 **37.** 03 **38.** 06 **39.** 03
40. 92 **41.** 29

EXERCISE - 1.5

1. 35 **2.** 18 **3.** 27 **4.** 51 **5.** 65 **6.** 15 **7.** 06
8. 93 **9.** 12 **10.** 25 **11.** 72 **12.** 02 **13.** 67 **14.** 23
15. 17 **16.** 02 **17.** 04 **18.** 05 **19.** 01 **20.** 25 **21.** 30
22. 37 **23.** 20 **24.** 40 **25.** 03

EXERCISE - 1.6

Que.	1	2	3	4	5	6	7	8	9
Ans.	1	1	4	4	1	3	3	1	2

10. 12 **11.** 32 **12.** 08 **13.** 02 **14.** 31 **15.** 36 **16.** 65
17. 16 **18.** 18 **19.** 12 **20.** 24 **21.** 25 **22.** 25 **23.** 58
24. 25 **25.** 03 **26.** 79 **27.** 35 **28.** 13 **29.** 06

EXERCISE - 1.7

Que.	1	2	3	4	5	6	7	8	9
Ans.	4	2	3	1	2	1	2	2	4

10. 49 **11.** 00 **12.** 28 **13.** 36 **14.** 48 **15.** 11 **16.** 03
17. 95 **18.** 24 **19.** 96 **20.** 50 **21.** 04 **22.** 20 **23.** 10
24. 27 **25.** 80 **26.** 52 **27.** 13 **28.** 31 **29.** 73 **30.** 25
31. 34 **32.** 11 **33.** 20 **34.** 05 **35.** 99 **36.** 53 **37.** 63
38. 37 **39.** 59 **40.** 03 **41.** 18 **42.** 14 **43.** 10 **44.** 78
45. 18 **46.** 09 **47.** 16 **48.** 99 **49.** 08 **50.** 23 **51.** 04
52. 17 **53.** 09 **54.** 04 **55.** 17 **56.** 09 **57.** 09

EXERCISE - 1.8

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	3	4	3	4	2	4	2	3	4	2

11. 67 **12.** 19 **13.** 10 **14.** 09 **15.** 14

EXERCISE - 1.9

Que.	1	2	3	4	5	6	7	8
Ans.	4	2	4	4	3	4	4	4

9. 21 **10.** 49 **11.** 59 **12.** 01 **13.** 02 **14.** 25 **15.** 90